TUW-02-16
YITP-02-35

The anomaly in the central charge of the supersymmetric kink from dimensional regularization and reduction

A. Rebhan[1a], P. van Nieuwenhuizen[2b] and R. Wimmer[1c]

[1] *Institut für Theoretische Physik, Technische Universität Wien, Wiedner Hauptstr. 8–10, A-1040 Vienna, Austria*

[2] *C.N.Yang Institute for Theoretical Physics, SUNY at Stony Brook, Stony Brook, NY 11794-3840, USA*

ABSTRACT

We show that the anomalous contribution to the central charge of the 1+1-dimensional N=1 supersymmetric kink that is required for BPS saturation at the quantum level can be linked to an analogous term in the extra momentum operator of a 2+1-dimensional kink domain wall with spontaneous parity violation and chiral domain wall fermions. In the quantization of the domain wall, BPS saturation is preserved by nonvanishing quantum corrections to the momentum density in the extra space dimension. Dimensional reduction from 2+1 to 1+1 dimensions preserves the unbroken $N=\frac{1}{2}$ supersymmetry and turns these parity-violating contributions into the anomaly of the central charge of the supersymmetric kink. On the other hand, standard dimensional regularization by dimensional reduction from 1 to $(1-\epsilon)$ spatial dimensions, which also preserves supersymmetry, locates the anomaly in an evanescent counterterm.

[a] rebhana@hep.itp.tuwien.ac.at
[b] vannieu@insti.physics.sunysb.edu
[c] rwimmer@hep.itp.tuwien.ac.at

1 Introduction

The calculation of quantum corrections to the mass of a supersymmetric (susy) kink and to its central charge has proved to be a highly nontrivial task fraught with subtleties and pitfalls.

Initially it was thought that supersymmetry would lead to a complete cancellation of quantum corrections [1] and thereby guarantee Bogomolnyi-Prasad-Sommerfield (BPS) saturation at the quantum level. Then, by considering a kink-antikink system in a finite box and regularizing the ultraviolet divergences by a cutoff in the number of the discretized modes, Schonfeld [2] found that there is a nonzero, negative quantum correction at one-loop level, $\Delta M^{(1)} = -m/(2\pi)$, but remarked that "the familiar sum of frequencies ... is unacceptably sensitive to the definition of the infinite volume limit". Most of the subsequent literature [3, 4, 5] considered instead a single kink directly, using an energy-momentum cutoff which gave again a null result. A direct calculation of the central charge [6] also gave a null result, apparently confirming a conjecture[1] of Witten and Olive [7] that BPS saturation in the minimally susy 1+1 dimensional case would hold although arguments on multiplet shortening do not seem to apply.

In Ref. [8] two of the present authors noticed a surprising dependence on the regularization method, even after the renormalization scheme has been fully fixed. In particular it was found that the naive energy-momentum cutoff as used in the susy case spoils the integrability of the bosonic sine-Gordon model [9]. Using a mode regularization scheme and periodic boundary conditions in a finite box instead led to a susy kink mass correction $\Delta M^{(1)} = +m(1/4 - 1/2\pi) > 0$ (obtained previously also in Ref. [10]) which together with the null result for the central charge appeared to be consistent with the BPS bound, but implying nonsaturation. Subsequently it was found by two of us together with Nastase and Stephanov [11] that the traditionally used periodic boundary conditions are questionable. Using instead topological boundary conditions which are invisible in the topological and in the trivial sector together with a "derivative regularization"[2] indeed led to a different result, namely that originally obtained by Schonfeld [2], which however appeared to be in conflict with the BPS inequality for a central charge without quantum corrections.

[1] "While we suspect that this is true we have no proof." [7]

[2] In mode regularization it turns out that one has to average over sets of boundary conditions to cancel both localized boundary energy and delocalized momentum [12, 13].

Since this appeared to be a pure one-loop effect, Ref. [11] proposed "...the interesting conjecture that it may be formulated in terms of a topological quantum anomaly." It was then shown by Shifman et al. [14], using a susy-preserving higher-derivative regularization method, that there is an anomalous contribution to the central charge balancing the quantum corrections to the mass so that BPS saturation remains intact. In fact, it was later understood that multiplet shortening does in fact occur even in minimally susy 1+1 dimensional theories, giving rise to single-state supermultiplets [15, 16].

Both results, the nonvanishing mass correction and thus the necessity of a nonvanishing correction to the central charge, have been confirmed by a number of different methods [17, 18, 12, 19, 20, 21, 13] validating also the finite mass formula in terms of only the discrete modes derived in Refs. [22, 23] based on the method of [24]. However, some authors claimed a nontrivial quantum correction to the central charge [17, 25] apparently without the need of the anomalous term proposed in Ref. [14].

In a previous paper [26], we have shown that a particularly simple and elegant regularization scheme that yields the correct quantum mass of the susy kink is dimensional regularization, if the kink is embedded in higher dimensions as a domain wall [27]. Such a scheme was not considered before for the susy kink because both susy and the existence of finite-energy solutions seemed to tie one to one spatial dimension.

In fact, the 1+1 dimensional susy kink can be embedded in 2+1 dimensions with the same field content while keeping susy invariance. For the corresponding classically BPS saturated domain wall (a 1+1 dimensional object by itself), we have found a nontrivial negative correction to the surface (i.e. string) tension [26]. In order to have BPS saturation at the quantum level, there has to be a matching correction to the momentum in the extra dimension which is the analog of the central charge of the 1+1 dimensional case.

In this work we show that in dimensional regularization by means of dimensional reduction from 2+1 dimensions, which preserves susy, one finds the required correction to the extra momentum to have a BPS saturated domain wall at the quantum level. This nontrivial correction is made possible by the fact that the 2+1 dimensional theory spontaneously breaks parity, which also allows the appearance of domain wall fermions of only one chirality.

By dimensionally reducing to 1+1 dimensions, the parity-violating contributions to the extra momentum turn out to provide an anomalous contribution to the central charge as postulated in Ref. [14], thereby giving a novel physical explanation of the latter. This is in line with the well-known fact that central charges of susy theories can be reinterpreted as momenta in higher dimensions.

In the case of the susy kink, dimensional regularization is seen to be compatible with susy invariance only at the expense of a spontaneous parity violation, which in turn allows nonvanishing quantum corrections to the extra momentum in one higher spatial dimension. On the other hand, the surface term that usually exclusively provides the central charge does not receive quantum corrections in dimensional regularization, by the same reason that led to null results previously in other schemes [6, 8, 11]. The nontrivial anomalous quantum correction to the central charge operator is thus seen to be entirely the remnant of the spontaneous parity violation in the higher-dimensional theory in which a susy kink can be embedded by preserving minimal susy.

2 Minimally supersymmetric kink and kink domain wall

2.1 The model

The real φ^4 model in 1+1 dimensions with spontaneously broken Z_2 symmetry ($\varphi \to -\varphi$) has topologically non-trivial finite-energy solutions called "kinks" which interpolate between the two degenerate vacuum states $\varphi = \pm v$. It has a minimally supersymmetric extension [28]

$$\mathcal{L} = -\frac{1}{2}\left[(\partial_\mu \varphi)^2 + U(\varphi)^2 + \bar{\psi}\gamma^\mu \partial_\mu \psi + U'(\varphi)\bar{\psi}\psi\right] \tag{1}$$

where ψ is a Majorana spinor, $\bar{\psi} = \psi^{\mathrm{T}} C$ with $C\gamma^\mu = -(\gamma^\mu)^T C$. We shall use a Majorana representation of the Dirac matrices with $\gamma^0 = -i\tau^2$, $\gamma^1 = \tau^3$, and $C = \tau^2$ in terms of the standard Pauli matrices τ^k so that $\psi = \binom{\psi^+}{\psi^-}$ with real $\psi^+(x,t)$ and $\psi^-(x,t)$.

The φ^4 model is defined as the special case

$$U(\varphi) = \sqrt{\frac{\lambda}{2}}\left(\varphi^2 - v_0^2\right), \qquad v_0^2 \equiv \mu_0^2/\lambda \tag{2}$$

where the Z_2 symmetry of the susy action also involves the fermions according to $\varphi \to -\varphi, \psi \to \gamma^5 \psi$ with $\gamma^5 = \gamma^0\gamma^1$. A classical kink at rest at $x = 0$ which interpolates between the two vacua $\varphi = \pm v_0$ is given by [29]

$$\varphi_K = v_0 \tanh\left(\mu_0 x/\sqrt{2}\right). \tag{3}$$

At the quantum level we have to renormalize, and we shall employ the simplest possible scheme[3] which consists of putting all renormalization constants to unity except for a mass counterterm chosen such that tadpole diagrams cancel completely in the trivial vacuum. At the one-loop level and using dimensional regularization this gives

$$\delta\mu^2 = \lambda\,\delta v^2 = \lambda \int \frac{dk_0 d^d k}{(2\pi)^{d+1}} \frac{-i}{k^2 + m^2 - i\epsilon} = \lambda \int \frac{d^d k}{(2\pi)^d} \frac{1}{2[\vec{k}^2 + m^2]^{1/2}}, \tag{4}$$

where $m = U'(v) = \sqrt{2}\mu$ is the mass of elementary bosons and fermions and $k^2 = \vec{k}^2 - k_0^2$.

The susy invariance of the model (with μ_0^2 replaced by $\mu^2 + \delta\mu^2$) leads to the on-shell conserved Noether current

$$j_\mu = -(\partial\!\!\!/\varphi + U(\varphi))\gamma_\mu \psi \tag{5}$$

and two conserved charges $Q^\pm = \int dx\, j_0^\pm$.

The model (1) is equally supersymmetric in 2+1 dimensions, where we use $\gamma^2 = \tau^1$. The same renormalization scheme can be used, only the renormalization constant (4) has to be evaluated for $d = 2 - \epsilon$ in place of $d = 1 - \epsilon$ spatial dimensions.

While classical kinks in 1+1 dimensions have finite energy (rest mass) $M = m^3/\lambda$, in (noncompact) 2+1 dimensions there exist no longer solitons of finite-energy. Instead one can have (one-dimensional) domain walls with a profile given by (3) which have finite surface (string) tension $M/L = m^3/\lambda$. With a compact extra dimension one can of course use these configurations to form "domain strings" of finite total energy proportional to the length L of the string when wrapped around the extra dimension.

The 2+1 dimensional case is different also with respect to the discrete symmetries of (1). In 2+1 dimensions, $\gamma^5 = \gamma^0\gamma^1\gamma^2 = \pm\mathbf{1}$ corresponding to the two inequivalent choices available for $\gamma^2 = \pm\tau^1$ (in odd space-time dimensions the Clifford algebra has two inequivalent irreducible representations). Therefore, the sign of the fermion mass (Yukawa) term can no

[3] See [26] for a detailed discussion of more general renormalization schemes in this context.

longer be reversed by $\psi \to \gamma^5\psi$ and there is no longer the Z_2 symmetry $\varphi \to -\varphi, \psi \to \gamma^5\psi$.

What the 2+1 dimensional model does break spontaneously is instead *parity*, which corresponds to changing the sign of one of the spatial coordinates. The Lagrangian is invariant under $x^m \to -x^m$ for a given spatial index $m = 1, 2$ together with $\varphi \to -\varphi$ (which thus is a pseudoscalar) and $\psi \to \gamma^m\psi$. Each of the trivial vacua breaks these invariances spontaneously, whereas a kink background in the x^1-direction with $\varphi_K(-x^1) = -\varphi_K(x^1)$ is symmetric with respect to x^1-reflections, but breaks $x^2 = y$ reflection invariance.

This is to be contrasted with the 1+1 dimensional case, where parity ($x^1 \to -x^1$) can be represented either by $\psi \to \gamma^0\psi$ and a true scalar $\varphi \to \varphi$ or by $\psi \to \gamma^1\psi$ and a pseudoscalar $\varphi \to -\varphi$. The former leaves the trivial vacuum invariant, and the latter the ground state of the kink sector.

2.2 Susy algebra

The susy algebra for the 1+1 and the 2+1 dimensional cases can both be covered by starting from 2+1 dimensions, the 1+1 dimensional case following from reduction by one spatial dimension.

In 2+1 dimensions one obtains classically [30]

$$\begin{aligned} \{Q^\alpha, \bar{Q}_\beta\} &= 2i(\gamma^M)^\alpha{}_\beta P_M \ , \quad (M = 0, 1, 2) \\ &= 2i(\gamma^0 H + \gamma^1(\tilde{P}_x + \tilde{Z}_y) + \gamma^2(\tilde{P}_y - \tilde{Z}_x))^\alpha{}_\beta, \end{aligned} \tag{6}$$

where we separated off two surface terms $\tilde{Z}_m$ in defining

$$\tilde{P}_m = \int d^dx \tilde{\mathcal{P}}_m, \qquad \tilde{\mathcal{P}}_m = \dot{\varphi}\, \partial_m \varphi - \frac{1}{2}(\bar{\psi}\gamma^0\partial_m\psi), \tag{7}$$

$$\tilde{Z}_m = \int d^dx \tilde{\mathcal{Z}}_m, \qquad \tilde{\mathcal{Z}}_m = U(\varphi)\partial_m\varphi = \partial_m W(\varphi) \tag{8}$$

with $W(\varphi) \equiv \int d\varphi\, U(\varphi)$.

Having a kink profile in the x-direction, which satisfies the Bogomolnyi equation $\partial_x\varphi_K = -U(\varphi_K)$, one finds that with our choice of Dirac matrices

$$Q^\pm = \int d^2x[(\dot{\varphi} \mp \partial_y\varphi)\psi^\pm + (\partial_x\varphi \pm U(\varphi))\psi^\mp], \tag{9}$$

$$\{Q^\pm, Q^\pm\} = 2(H \pm (\tilde{Z}_x - \tilde{P}_y)), \tag{10}$$

and the charge Q^+ leaves the topological (domain-wall) vacuum $\varphi = \varphi_K$, $\psi = 0$ invariant. This corresponds to classical BPS saturation, since with $P_x = 0$ and $\tilde{P}_y = 0$ one has $\{Q^+, Q^+\} = 2(H + \tilde{Z}_x)$ and, indeed, with a kink domain wall $\tilde{Z}_x/L^{d-1} = W(+v) - W(-v) = -M/L^{d-1}$.

At the quantum level, hermiticity of $Q^\pm$ implies

$$\langle s|H|s\rangle \geq |\langle s|P_y|s\rangle| \equiv |\langle s|(\tilde{P}_y - \tilde{Z}_x)|s\rangle|. \tag{11}$$

This inequality is saturated when

$$Q^+|s\rangle = 0 \tag{12}$$

so that BPS states correspond to massless states $P_M P^M = 0$ with $P_y = M$ for a kink domain wall in the x-direction [16], however with infinite momentum and energy unless the y-direction is compact with finite length L. An antikink domain wall has instead $Q^-|s\rangle = 0$. In both cases, half of the supersymmetry is spontaneously broken.

Classically, the susy algebra in 1+1 dimensions is obtained from (6) simply by dropping $\tilde{P}_y$ as well as $\tilde{Z}_y$ so that $P_x \equiv \tilde{P}_x$. The term $\gamma^2 \tilde{Z}_x$ remains, however, with γ^2 being the nontrivial γ^5 of 1+1 dimensions. The susy algebra simplifies to

$$\{Q^\pm, Q^\pm\} = 2(H \pm Z), \quad \{Q^+, Q^-\} = 2P_x \tag{13}$$

and one has the inequality

$$\langle s|H|s\rangle \geq |\langle s|Z|s\rangle| \tag{14}$$

for any quantum state s. BPS saturated states have $Q^+|s\rangle = 0$ or $Q^-|s\rangle = 0$, corresponding to kink and antikink, respectively, and break half of the supersymmetry.

3 Quantum corrections to the susy algebra

3.1 Fluctuations

In a kink (or kink domain wall) background one spatial direction is singled out and we choose this to be along x. The direction orthogonal to the kink direction (parallel to the domain wall) will be denoted by y.

The quantum fields can then be expanded in the analytically known kink eigenfunctions [29] times plane waves in the extra dimensions. For the bosonic fluctuations we have $[-\Box + (U'^2 + UU'')]\eta = 0$ which is solved by

$$\eta = \int \frac{d^{d-1}\ell}{(2\pi)^{\frac{d-1}{2}}} \sum\!\!\!\!\!\!\!\int \frac{dk}{\sqrt{4\pi\omega}} \left(a_{k,\ell}\; e^{-i(\omega t - \ell y)} \phi_k(x) + a^\dagger_{k,\ell}\; e^{i(\omega t - \ell y)} \phi^*_k(x) \right). \quad (15)$$

The kink eigenfunctions ϕ_k are normalized according to $\int dx |\phi|^2 = 1$ for the discrete states and to Dirac distributions for the continuum states according to $\int dx\, \phi^*_k \phi_{k'} = 2\pi\delta(k - k')$. The mode energies are $\omega = \sqrt{\omega_k^2 + \ell^2}$ where ω_k is the energy in the 1+1-dimensional case.

The canonical equal-time commutation relations $[\eta(\vec{x}), \dot{\eta}(\vec{x}')] = i\delta(\vec{x} - \vec{x}')$ are fulfilled with

$$[a_{k,\ell}, a^\dagger_{k',\ell'}] = \delta_{kk'}\delta(\ell - \ell'), \quad (16)$$

where for the continuum states $\delta_{k,k'}$ becomes a Dirac delta.

For the fermionic modes which satisfy the Dirac equation $[\not{\partial} + U']\psi = 0$ one finds

$$\begin{aligned} \psi &= \psi_0 + \int \frac{d^{d-1}\ell}{(2\pi)^{\frac{d-1}{2}}} \sum\!\!\!\!\!\!\!\int{}' \frac{dk}{\sqrt{4\pi\omega}} \left[b_{k,\ell}\; e^{-i(\omega t - \ell y)} \begin{pmatrix} \sqrt{\omega + \ell}\; \phi_k(x) \\ \sqrt{\omega - \ell}\; i s_k(x) \end{pmatrix} + b^\dagger_{k,\ell}\; (c.c.) \right], \\ \psi_0 &= \int \frac{d^{d-1}\ell}{(2\pi)^{\frac{d-1}{2}}} b_{0,\ell}\; e^{-i\ell(t-y)} \begin{pmatrix} \phi_0 \\ 0 \end{pmatrix}, \quad b_0^\dagger(\ell) = b_0(-\ell). \end{aligned} \quad (17)$$

The fermionic zero mode[4] of the susy kink turns into massless modes located on the domain wall, which have only one chirality, forming a Majorana-Weyl domain wall fermion [26, 31].[5]

For the massive modes the Dirac equation relates the eigenfunctions appearing in the upper and the lower components of the spinors as follows:

$$s_k = \frac{1}{\omega_k}(\partial_x + U')\phi_k = \frac{1}{\sqrt{\omega^2 - \ell^2}}(\partial_x + U')\phi_k, \quad (18)$$

so that the function s_k is the SUSY-quantum mechanical [32] partner state of ϕ_k and thus coincides with the eigen modes of the sine-Gordon model (hence

[4]By a slight abuse of notation we shall always label this by a subscript 0, but this should not be confused with the threshold mode $k = 0$ (which does not appear explicitly anywhere below).

[5]The mode with $\ell = 0$ corresponds in 1+1 dimensions to the zero mode of the susy kink. It has to be counted as half a degree of freedom in mode regularization [12]. For dimensional regularization such subtleties do not play a role because the zero mode only gives scaleless integrals and these vanish.

the notation) [33]. With (18), their normalization is the same as that of the ϕ_k.

The canonical equal-time anti-commutation relations $\{\psi^\alpha(\vec{x}), \psi^\beta(\vec{x}')\} = \delta^{\alpha\beta}\delta(\vec{x}-\vec{x}')$ are satisfied if

$$\begin{aligned} \{b_0(\ell), b_0^\dagger(\ell')\} &= \{b_0(\ell), b_0(-\ell')\} = \delta(\ell-\ell'), \\ \{b_{k,\ell}, b_{k',\ell'}^\dagger\} &= \delta_{k,k'}\delta(\ell-\ell'), \end{aligned} \tag{19}$$

and again the $\delta_{k,k'}$ becomes a Dirac delta for the continuum states. The algebra (19) and the solution for the massless mode (17) show that the operator $b_0(\ell)$ creates right-moving massless states on the wall when ℓ is negative and annihilates them for positive momentum ℓ. Thus only massless states with momentum in the positive y-direction can be created. Changing the representation of the gamma matrices by $\gamma^2 \to -\gamma^2$, which is inequivalent to the original one, reverses the situation. Now only massless states with momenta in the positive y-direction exist. Thus depending on the representation of the Clifford algebra one chirality of the domain wall fermions is singled out. This is a reflection of the spontaneous violation of parity when embedding the susy kink as a domain wall in 2+1 dimensions.

Notice that in (17) d can be only 2 or 1, for which ℓ has 1 or 0 components, so for strictly $d = 1$ $\ell \equiv 0$. In order to have a susy-preserving dimensional regularization scheme by dimensional reduction, we shall start from $d = 2$ spatial dimensions, and then make d continuous and smaller than 2.

3.2 Energy corrections

Before turning to a direct calculation of the anomalous contributions to central charge and momentum, we recapitulate the one-loop calculation of the energy density of the susy kink (domain wall) in dimensional regularization.

Expanding the Hamiltonian density of the model (1),

$$\mathcal{H} = \frac{1}{2}[\dot{\varphi} + (\nabla\varphi)^2 + U^2(\varphi)] + \frac{1}{2}\psi^\dagger i\gamma^0[\vec{\gamma}\,\vec{\nabla} + U'(\varphi)]\psi, \tag{20}$$

around the kink/domain wall, using $\varphi = \varphi_K + \eta$, one obtains

$$\begin{aligned} \mathcal{H} &= \frac{1}{2}[(\partial_x\varphi_K)^2 + U^2] - \frac{\delta\mu^2}{\sqrt{2\lambda}}U - \partial_x(U\eta) + \\ &\quad + \frac{1}{2}[\dot{\eta}^2 + (\nabla\eta)^2 + \frac{1}{2}(U^2)''\eta^2] + \frac{1}{2}\psi^\dagger i\gamma^0[\vec{\gamma}\,\vec{\nabla} + U']\psi + O(\hbar^2), \end{aligned} \tag{21}$$

where U without an explicit argument implies evaluation at $\varphi = \varphi_K$ and use of the renormalized μ^2. The first two terms on the r.h.s. are the classical energy density and the counterterm contribution. The terms quadratic in the fluctuations are the only ones contributing to the total energy.[6] They give

$$\begin{aligned}
\int dx\, d^{d-1}y\, \langle \mathcal{H}^{(2)} \rangle \;=\;& \frac{L^{d-1}}{2} \int dx \int \frac{d^{d-1}\ell}{(2\pi)^{d-1}} \sum\!\!\!\!\!\!\int \frac{dk}{2\pi} \Big[\frac{\omega}{2} |\phi_k|^2 \\
&+ \frac{1}{2\omega} (\ell^2 |\phi_k|^2 + |\phi_k{}'|^2 + \frac{1}{2} (U^2)'' |\phi_k|^2) \Big] \\
-\;& \frac{L^{d-1}}{2} \int dx \int \frac{d^{d-1}\ell}{(2\pi)^{d-1}} \sum\!\!\!\!\!\!\int \frac{dk}{2\pi} \frac{\omega}{2} (|\phi_k|^2 + |s_k|^2), \quad (22)
\end{aligned}$$

where the two sum-integrals are the bosonic and fermionic contributions, respectively.

Using $\frac{1}{2}(U^2)'' = U'^2 - \partial_x U'$ which follows from the Bogomolnyi equation $\partial_x \varphi_K = -U$ and partially integrating (or alternatively from the equipartition theorem for the energy of the bosonic fluctuations in (21)), one obtains the expected sum-integrals over zero-point energies,

$$\int dx\, d^{d-1}y\, \langle \mathcal{H}^{(2)} \rangle = \frac{L^{d-1}}{2} \int dx \int \frac{d^{d-1}\ell}{(2\pi)^{d-1}} \sum\!\!\!\!\!\!\int \frac{dk}{2\pi} \frac{\omega}{2} (|\phi_k|^2 - |s_k|^2). \qquad (23)$$

In these expressions, the massless modes (which correspond to the zero mode of the 1+1 dimensional kink) can be dropped in dimensional regularization as scaleless and thus vanishing contributions, and the massive discrete modes cancel between bosons and fermions.[7] Carrying out the x-integration over the continuous mode functions gives a difference of spectral densities, namely

$$\int dx (|\phi_k(x)|^2 - |s_k(x)|^2) = -\theta'(k) = -\frac{2m}{k^2 + m^2}, \qquad (24)$$

where $\theta(k)$ is the additional phase shift of the mode functions s_k compared to ϕ_k.

[6] The third term in (21) is of relevance when calculating the energy profile [14, 19].

[7] The zero mode contributions in fact do not cancel by themselves between bosons and fermions, because the latter are chiral. This noncancellation is in fact crucial in energy cutoff regularization (see Ref. [26]).

Combining (23) and (24), and adding in the counterterm contribution from (4) leads to a simple integral

$$\begin{aligned}\frac{\Delta M^{(1)}}{L^{d-1}} &= -\frac{1}{4}\int \frac{dk\, d^{d-1}\ell}{(2\pi)^d}\omega\,\theta'(k) + m\delta v^2 \\ &= -\frac{1}{4}\int \frac{dk\, d^{d-1}\ell}{(2\pi)^d}\frac{\ell^2}{\omega}\theta'(k) = -\frac{2}{d}\frac{\Gamma(\frac{3-d}{2})}{(4\pi)^{\frac{d+1}{2}}}\, m^d. \end{aligned} \tag{25}$$

This reproduces the correct known result for the susy kink mass correction $\Delta M^{(1)} = -m/(2\pi)$ (for $d = 1$) and the surface (string) tension of the 2+1 dimensional susy kink domain wall $\Delta M^{(1)}/L = -m^2/(8\pi)$ (for $d = 2$) [26].

Notice that the entire result is produced by an integrand proportional to the extra momentum component ℓ^2, which for strictly $d = 1$ would not exist. This can also be observed by recasting $\langle\mathcal{H}^{(2)}\rangle$ in (22) with the help of (18) in the form

$$\begin{aligned}\langle\mathcal{H}^{(2)}\rangle &= -\partial_x\left(\frac{1}{2}\int\frac{d^{d-1}\ell}{(2\pi)^{d-1}}\sum\!\!\!\!\!\!\int\frac{dk}{2\pi}\,U'\frac{|\phi_k|^2}{2\omega}\right) + \\ &\quad + \frac{1}{2}\int\frac{d^{d-1}\ell}{(2\pi)^{d-1}}\sum\!\!\!\!\!\!\int\frac{dk}{2\pi}\frac{\ell^2}{2\omega}(|\phi_k^2| - |s_k|^2). \end{aligned} \tag{26}$$

When integrated, the first term, which is a pure surface term, cancels exactly the counterterm (see (4)), because

$$\int dx\langle\frac{1}{2}\partial_x(U'\eta^2)\rangle = \frac{1}{2}U'\langle\eta^2\rangle|_{-\infty}^{\infty} = m\int\frac{d^{d-1}\ell}{(2\pi)^{d-1}}\int\frac{dk}{2\pi}\frac{1}{2\omega} \equiv m\delta v^2, \tag{27}$$

where we have used that $U'(x = \pm\infty) = \pm m$. The second contribution in (26), on the other hand, is precisely the r.h.s. of (25).

3.3 Anomalous contributions to the central charge and extra momentum

In a kink (domain wall) background with only nontrivial x dependence, the central charge density $\tilde{\mathcal{Z}}_x$ receives nontrivial contributions. Expanding $\tilde{\mathcal{Z}}_x$ around the kink background gives

$$\tilde{\mathcal{Z}}_x = U\partial_x\varphi_K - \frac{\delta\mu^2}{\sqrt{2\lambda}}\partial_x\varphi_K + \partial_x(U\eta) + \frac{1}{2}\partial_x(U'\eta^2) + O(\eta^3). \tag{28}$$

Again only the part quadratic in the fluctuations contributes to the integrated quantity at one-loop order[8]. However, this leads just to the contribution shown in (27), which matches precisely the counterterm $m\delta v^2$ from requiring vanishing tadpoles. Straightforward application of the rules of dimensional regularization thus leads to a null result for the net one-loop correction to $\langle \tilde{Z}_x \rangle$ in the same way as found in Refs. [6, 8, 11] in other schemes.

On the other hand, by considering the less singular combination $\langle H + \tilde{Z}_x \rangle$ and showing that it vanishes exactly, it was concluded in Ref. [17] that $\langle \tilde{Z}_x \rangle$ has to compensate any nontrivial result for $\langle H \rangle$, which in Ref. [17] was obtained by subtracting successive Born approximations for scattering phase shifts. In fact, Ref. [17] explicitly demonstrates how to rewrite $\langle \tilde{Z}_x \rangle$ into $-\langle H \rangle$, apparently without the need for the anomalous terms in the quantum central charge operator postulated in Ref. [14].

The resolution of this discrepancy is that Ref. [17] did not regularize $\langle \tilde{Z}_x \rangle$ and the manipulations needed to rewrite it as $-\langle H \rangle$ (which eventually is regularized and renormalized) are ill-defined. Using dimensional regularization one in fact obtains a nonzero result for $\langle H + \tilde{Z}_x \rangle$, apparently in violation of susy.

However, dimensional regularization by embedding the kink as a domain wall in (up to) one higher dimension, which preserves susy, instead leads to

$$\langle H + \tilde{Z}_x - \tilde{P}_y \rangle = 0, \tag{29}$$

i.e. the saturation of (11), as we shall now verify.

The bosonic contribution to $\langle \tilde{P}_y \rangle$ involves

$$\frac{1}{2}\langle \dot{\eta}\partial_y \eta + \partial_y \eta \dot{\eta} \rangle = -\int \frac{d^{d-1}\ell}{(2\pi)^{d-1}} \sum\!\!\!\!\!\!\int \frac{dk}{2\pi}\, \frac{\ell}{2} |\phi_k(x)|^2. \tag{30}$$

The ℓ-integral factorizes and gives zero both because it is a scale-less integral and because the integrand is odd in ℓ. Only the fermions turn out to give interesting contributions:

$$\begin{aligned}
\langle \tilde{\mathcal{P}}_y \rangle &= \frac{i}{2}\langle \psi^\dagger \partial_y \psi \rangle \\
&= \frac{1}{2}\int \frac{d^{d-1}\ell}{(2\pi)^{d-1}} \sum\!\!\!\!\!\!\int \frac{dk}{2\pi}\frac{\ell}{2\omega}\left[(\omega+\ell)|\phi_k|^2 + (\omega-\ell)|s_k|^2\right]
\end{aligned}$$

[8]Again, this does not hold for the central charge density locally [14, 19].

$$
\begin{aligned}
&= \frac{1}{2}\int \frac{d^{d-1}\ell}{(2\pi)^{d-1}}\, \ell\, \theta(-\ell)\, |\phi_0|^2 + \\
&+\frac{1}{2}\int \frac{d^{d-1}\ell}{(2\pi)^{d-1}} \sumint{}' \frac{dk}{2\pi}\left(\frac{\ell}{2}(|\phi_k|^2+|s_k|^2)+\frac{\ell^2}{2\omega}(|\phi_k|^2-|s_k|^2)\right). \quad (31)
\end{aligned}
$$

From the last sum-integral we have separated off the contribution of the zero mode of the kink, which turns into chiral domain wall fermions for $d > 1$. The contribution of the latter no longer vanishes by symmetry, but the ℓ-integral is still scale-less and therefore put to zero in dimensional regularization. The first sum-integral on the right-hand side is again zero by both symmetry and scalelessness, but the final term is not. The ℓ-integration no longer factorizes because $\omega = \sqrt{k^2 + \ell^2 + m^2}$, and is in fact identical to the finite contribution in $\langle \mathcal{H} \rangle$ obtained already in (26).

So for all $d \leq 2$ we have BPS saturation, $\langle H \rangle = |\langle \tilde{Z}_x - \tilde{P}_y \rangle|$, which in the limit $d \to 1$, the susy kink, is made possible by a nonvanishing $\langle \tilde{P}_y \rangle$. The anomaly in the central charge is seen to arise from a parity-violating contribution in $d = 1 + \epsilon$ dimensions which is the price to be paid for preserving supersymmetry when going up in dimensions to embed the susy kink as a domain wall.

It is perhaps worth emphasizing that the above results do not depend on the details of the spectral densities associated with the mode functions ϕ_k and s_k. In the integrated quantities $\langle H \rangle$ and $\langle \tilde{P}_y \rangle$ only the difference of the spectral densities as given by (24) is responsible for the nonvanishing contribution. The function $\theta(k)$ therein is entirely fixed by the form of the Dirac equation in the asymptotic regions $x \to \pm\infty$ far away from the kink [8].

4 Discussion

For the susy kink, we have effectively used the 't Hooft-Veltman version of dimensional regularization [34] in which the space-time dimensionality n is made larger than the dimension of interest. In general this breaks susy because the numbers of bosons and fermions are not the same anymore when one moves up in dimensions. But in our particular model the number of states are the same in 1+1 and 2+1 dimensions, so that we could preserve susy, though this led to new physics like spontaneous parity violation and chiral domain wall fermions.

In 2+1 dimensions, we have $P_y = \tilde{P}_y - \tilde{Z}_x$ and $|\langle P_y \rangle| = \langle H \rangle$. Classically, this BPS saturation is guaranteed by $\tilde{Z}_x$ alone. At the quantum level, however, the quantum corrections to the latter are cancelled completely by the counterterm from renormalizing tadpoles to zero. All nontrivial corrections come from the "genuine" momentum operator $\tilde{P}_y$, and are due to having a spontaneous breaking of parity.

In the limit of 1+1 dimensions, it is natural to make the identification $Z = \tilde{Z}_x - \tilde{P}_y$. For $\tilde{Z}_x$, one again does not obtain net quantum corrections. However, the expectation value $\langle \tilde{P}_y \rangle$ does not vanish in the limit $d \to 1$, although there is no longer an extra dimension. The spontaneous parity violation in the 2+1 dimensional theory, which had to be considered in order to preserve susy, leaves a finite imprint upon dimensional reduction to 1+1 dimensions by providing an anomalous additional contribution to $\langle \tilde{Z}_x \rangle$ balancing the nontrivial quantum correction $\langle H \rangle$.

We now comment on how the central charge anomaly may be recovered from Siegel's version of dimensional regularization [35] where n is smaller than the dimension of spacetime and where one keeps the number of field components fixed, but lowers the number of coordinates and momenta from 2 to $n < 2$. At the one-loop level one encounters 2-dimensional δ_μ^ν coming from Dirac matrices, and n-dimensional $\hat{\delta}_\mu^\nu$ from loop momenta. An important concept which is going to play a role are the evanescent counterterms [36] such as $\frac{1}{\epsilon}\hat{\hat{\delta}}_\mu^\nu \gamma_\nu \psi$, where $\hat{\hat{\delta}}_\mu^\nu \equiv \delta_\mu^\nu - \hat{\delta}_\mu^\nu$ has only $\epsilon = 2 - n$ nonvanishing components.

For the chiral anomaly in 3+1 dimensions due to a massless Dirac fermion coupled to on-shell photons one finds from dimensional reduction the following expression for the regularized but not yet renormalized chiral current [37]

$$j_\mu = \frac{1}{2}\partial_\mu \frac{1}{\Box} F^{\rho\sigma}\tilde{F}_{\rho\sigma} - \frac{2}{\epsilon}\tilde{F}_{\mu\nu}\hat{\hat{A}}^\nu \tag{32}$$

where $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$ and $\hat{\hat{A}}^\nu \equiv \hat{\hat{\delta}}_\mu^\nu A^\mu$. This current is gauge invariant because $\delta\hat{\hat{A}}_\nu = \hat{\hat{\partial}}_\nu \lambda = 0$ as coordinates only lie in the n-dimensional subspace. It is conserved since total antisymmetrization of five indices in 4 dimensions yields

$$\tilde{F}^{\mu\nu}\partial_\mu \hat{\hat{A}}_\nu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}\partial_\mu \hat{\hat{\delta}}_\nu^\lambda A_\lambda = -\tilde{F}^{\rho\nu}\partial_\rho \hat{\hat{A}}_\nu + \epsilon\frac{1}{2}F^{\mu\nu}\tilde{F}_{\mu\nu} \tag{33}$$

Clearly, $\partial^\mu j_\mu = \frac{1}{2}F^{\mu\nu}\tilde{F}_{\mu\nu} - \frac{1}{\epsilon}(\frac{1}{2}\epsilon F^{\mu\nu}\tilde{F}_{\mu\nu}) = 0$. The composite operator j_μ is renormalized by subtracting the divergence $-\frac{1}{\epsilon}\tilde{F}_{\mu\nu}\hat{\hat{A}}^\nu$ (operator mixing), and thus in dimensional reduction the chiral anomaly is produced by the (evanescent) counterterm, and not by the loop graph.

Consider now the supercurrent $j_\mu = -(\partial\!\!\!/\varphi + U(\varphi))\gamma_\mu\psi$. In the trivial vacuum, expanding into quantum fields yields

$$j_\mu = -\left(\partial\!\!\!/\eta + U'(v)\eta + \sqrt{\frac{\lambda}{2}}\eta^2\right)\gamma_\mu\psi + \frac{1}{\sqrt{2\lambda}}\delta\mu^2\gamma_\mu\psi. \tag{34}$$

Only matrix elements with one external fermion are divergent. The η-tadpole is cancelled by the counterterm $\delta\mu^2$ (which itself is due to an η and a ψ loop) so that only the graph of the form of a ψ-selfenergy remains. One finds[9] that the divergence in j_μ is proportional to the evanescent operator $\frac{1}{\epsilon}\hat{\hat{\delta}}{}^\lambda_\mu\gamma_\lambda\psi$. The current should again be without anomaly, but after renormalizing j_μ by subtracting the evanescent counterterm, the conformal-susy anomaly in j_μ is nonvanishing. Repeating this calculation in the presence of a kink yields the anomalous contribution to the central charge.

References

[1] A. D'Adda, P. Di Vecchia, Phys. Lett. B73 (1978) 162;
A. D'Adda, R. Horsley, P. Di Vecchia, Phys. Lett. B76 (1978) 298;
R. Horsley, Nucl. Phys. B151 (1979) 399.

[2] J. F. Schonfeld, Nucl. Phys. B161 (1979) 125.

[3] R. K. Kaul, R. Rajaraman, Phys. Lett. B131 (1983) 357.

[4] A. Chatterjee, P. Majumdar, Phys. Rev. D30 (1984) 844; Phys. Lett. B159 (1985) 37.

[5] H. Yamagishi, Phys. Lett. B147 (1984) 425.

[6] C. Imbimbo, S. Mukhi, Nucl. Phys. B247 (1984) 471.

[7] E. Witten, D. Olive, Phys. Lett. B78 (1978) 97.

[8] A. Rebhan, P. van Nieuwenhuizen, Nucl. Phys. B508 (1997) 449.

[9] R. F. Dashen, B. Hasslacher, A. Neveu, Phys. Rev. D11 (1975) 3424.

[9] Use $\kappa\!\!\!/\gamma_\mu\,\kappa\!\!\!/ = -\kappa^2(\delta^\lambda_\mu - \frac{2}{n}\hat{\delta}^\lambda_\mu)\gamma_\lambda$ and $\hat{\delta}^\nu_\mu \equiv \delta^\nu_\mu - \hat{\hat{\delta}}{}^\nu_\mu$ to obtain $-\kappa^2(1-\frac{2}{n})\delta^\lambda_\mu + \kappa^2\frac{2}{n}\hat{\hat{\delta}}{}^\lambda_\mu$.

[10] A. Uchiyama, Prog. Theor. Phys. 75 (1986) 1214.

[11] H. Nastase, M. Stephanov, P. van Nieuwenhuizen, A. Rebhan, Nucl. Phys. B542 (1999) 471.

[12] A. S. Goldhaber, A. Litvintsev, P. van Nieuwenhuizen, Phys. Rev. D64 (2001) 045013.

[13] A. S. Goldhaber, A. Rebhan, P. van Nieuwenhuizen, R. Wimmer, Clash of discrete symmetries for the supersymmetric kink on a circle, hep-th/0206229.

[14] M. A. Shifman, A. I. Vainshtein, M. B. Voloshin, Phys. Rev. D59 (1999) 045016.

[15] A. Losev, M. A. Shifman, A. I. Vainshtein, Phys. Lett. B522 (2001) 327.

[16] A. Losev, M. A. Shifman, A. I. Vainshtein, New Jour. Phys. 4 (2002) 21.

[17] N. Graham, R. L. Jaffe, Nucl. Phys. B544 (1999) 432.

[18] A. Litvintsev, P. van Nieuwenhuizen, Once more on the BPS bound for the susy kink, hep-th/0010051.

[19] A. S. Goldhaber, A. Litvintsev, P. van Nieuwenhuizen, Local Casimir energy for solitons, hep-th/0109110.

[20] R. Wimmer, Quantization of supersymmetric solitons, hep-th/0109119.

[21] M. Bordag, A. S. Goldhaber, P. van Nieuwenhuizen, D. Vassilevich, Heat kernels and zeta-function regularization for the mass of the susy kink, hep-th/0203066.

[22] J. Casahorrán, J. Phys. A22 (1989) L413.

[23] L. J. Boya, J. Casahorrán, J. Phys. A23 (1990) 1645.

[24] K. Cahill, A. Comtet, R. J. Glauber, Phys. Lett. B64 (1976) 283.

[25] J. Casahorrán, J. Phys. A22 (1989) L1167.

[26] A. Rebhan, P. van Nieuwenhuizen, R. Wimmer, New Jour. Phys. 4 (2002) 31.

[27] C. G. Bollini, J. J. Giambiagi, Dimensional regularization and finite temperature divergent determinants, CBPF-NF-085/83;
E. Brézin, S. Feng, Phys. Rev. B29 (1984) 472;
G. Münster, Nucl. Phys. B324 (1989) 630;
A. Parnachev, L. G. Yaffe, Phys. Rev. D62 (2000) 105034.

[28] P. Di Vecchia, S. Ferrara, Nucl. Phys. B130 (1977) 93.

[29] R. Rajaraman, Solitons and Instantons, North-Holland, Amsterdam, 1982.

[30] G. W. Gibbons, P. K. Townsend, Phys. Rev. Lett. 83 (1999) 1727.

[31] C. G. Callan, J. A. Harvey, Nucl. Phys. B250 (1985) 427;
G. W. Gibbons, N. D. Lambert, Phys. Lett. B488 (2000) 90;
R. Hofmann, T. ter Veldhuis, Phys. Rev. D63 (2001) 025017;
C. D. Fosco, Quantum stability of defects for a Dirac field coupled to a scalar field in 2+1 dimensions, hep-th/0205312.

[32] E. Witten, Nucl. Phys. B202 (1982) 253.

[33] F. Cooper, A. Khare, U. Sukhatme, Supersymmetry in Quantum Mechanics, World Scientific, Singapore, 2001.

[34] G. 't Hooft, M. Veltman, Nucl. Phys. B44 (1972) 189.

[35] W. Siegel, Phys. Lett. B84 (1979) 193; Phys. Lett. B94 (1980) 37;
D. M. Capper, D. R. T. Jones, P. van Nieuwenhuizen, Nucl. Phys. B167 (1980) 479.

[36] G. Bonneau, Nucl. Phys. B167 (1980) 261; Nucl. Phys. B171 (1980) 477.

[37] M. T. Grisaru, B. Milewski, D. Zanon, The supercurrent and the Adler-Bardeen theorem, in: G. W. Gibbons, S. W. Hawking, P. K. Townsend (Eds.), Supersymmetry and its applications: superstrings, anomalies and supergravity, Cambridge Univ. Press, Cambridge, 1986, pp. 55–62; Phys. Lett. B157 (1985) 174; Nucl. Phys. B266 (1986) 589.